JOURNAL OF APPROXIMATION THEORY 72, 237–251 (1993)

Mean Convergence of Generalized Jacobi Series and Interpolating Polynomials, I

YUAN XU*

Department of Mathematics, The University of Texas at Austin, Austin, Texas 78712, U.S.A.

Communicated by Paul Nevai

Received July 19, 1990; accepted in revised form December 31, 1991

Weighted mean convergence of generalized Jacobi series is investigated, and the results are used to prove weighted mean convergence of various interpolating polynomials based on the zeros of generalized Jacobi polynomials.

1. INTRODUCTION

The purpose of this paper is to investigate weighted mean convergence of the generalized Jacobi series, and weighted mean convergence of interpolating polynomials based on the zeros of the generalized Jacobi polynomials.

The generalized Jacobi series is the Fourier series of the generalized Jacobi polynomials, $p_n(w, x)$, which are the orthogonal polynomials in $[-1, 1]$ associated with weight functions of the form $w(x) = \prod_{i=0}^{r+1} |x - t_i|^{\Gamma_i}$, where $-1 = t_0 < t_1 < \cdots < t_{r+1} = 1$ and $\Gamma_i > -1$. These orthogonal polynomials have been studied extensively in [2, 14]. In the special cases $\Gamma_i = 0$, $1 \leqslant i \leqslant r$, they reduce to the classical Jacobi polynomials. In order to prove the mean convergence of the generalized Jacobi series, we will prove the following inequality

$$\|S_n(f)\, U\|_p \leqslant c\, \|fV\|_p, \tag{1.1}$$

where $S_n(f)$ is the nth partial sum of the generalized Jacobi series of f, U and V are suitable weight functions, and c is a constant independent of n and f. When $U = V$, the inequality (1.1) has been studied by Badkov [2], while in the special cases of the Jacobi polynomials it was considered by Pollard [19] and Muckenhoupt [11]. Our consideration of different U

* Current address: Department of Mathematics, University of Oregon, Eugene, OR 97403-1222.

0021-9045/93 $5.00

and V in (1.1) is motivated by the close relation between the weighted mean convergence of the Fourier orthogonal series and the weighted mean convergence of the Lagrange interpolating polynomials. This relation, first discovered by Marcinkiewicz and Zygmund [7, 8] in investigating the mean convergence of the trigonometric interpolation, enables one to reduce the mean convergence of the Lagrange interpolating polynomials based on the zeros of the orthogonal polynomials to that of orthogonal series. The idea of Marcinkiewicz was partly incorporated in the work of Askey [1], and later, of Nevai [14, 15]. Recently, we found in [28] that the full strength of Marcinkiewicz's method can be extended to Lagrange interpolation based on the zeros of the Jacobi polynomials. With the help of (1.1), we will further extend this method to the cases of the generalized Jacobi polynomials. The aim is the inequality of the type

$$\int_{-1}^{1} |P|^p w\, dx \leqslant c \sum_{k=1}^{n} |P(x_{kn})|^p \lambda_{kn}, \qquad 1<p<+\infty, \tag{1.2}$$

where P is a polynomial of degree at most $n-1$, x_{kn} are the zeros of $p_n(w, x)$, c is a constant independent of n and P. The inequality of this type is called the Marcinkiewicz–Zygmund inequality. Moreover, we shall extend this inequality to include the derivative values of P in the right hand side. The latter extension will enable us to deal with the mean convergence of Hermite interpolation.

The paper is organized as follows. In the next section, we shall give the definitions and list the basic facts about the generalized Jacobi polynomials. We prove some general inequalities in Section 3, and apply them to prove the inequalities of type (1.1) and the mean convergence of the generalized Jacobi series in Section 4. The Marcinkiewicz–Zygmund inequality, its extension, and the mean convergence of interpolating polynomials are the contents of Part II.

2. Preliminaries

Let $d\alpha = \alpha'(x)\, dx$ be a nonnegative distribution on $[-1, 1]$. Let $p_n(d\alpha, x)$ be the sequence of polynomials orthonormal with respect to $d\alpha$. The zeros of $p_n(d\alpha)$ are denoted by $x_{kn}(d\alpha)$ and the following order is assumed

$$1 > x_{1n}(d\alpha) > x_{2n}(d\alpha) > \cdots > x_{nn}(d\alpha) > -1. \tag{2.1}$$

The reproducing kernel functions of the orthogonal system $\{p_n(d\alpha)\}$ are denoted by $K_n(d\alpha)$,

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_k(d\alpha, t). \tag{2.2}$$

The Christoffel function $\lambda_n(d\alpha)$ is defined by

$$\lambda_n(d\alpha, x) = K_n(d\alpha, x, x)^{-1}. \tag{2.3}$$

The numbers $\lambda_{kn}(d\alpha) = \lambda_n(d\alpha, x_{kn})$ are called the Cotes numbers. By the Gauss–Jacobi quadrature formula [22, p. 47],

$$\sum_{k=1}^{n} P(x_{kn}(d\alpha))\,\lambda_{kn}(d\alpha) = \int_{-1}^{1} P\,d\alpha \tag{2.4}$$

holds for every polynomial $P \in \Pi_{2n-1}$. Here, Π_n is the space of polynomials of degree at most n.

If $0 < p \leqslant +\infty$, then $f \in L^p$ if $\|f\|_p < +\infty$, where

$$\|f\|_p = \left(\int_{-1}^{1} |f(t)|^p\,dt\right)^{1/p}, \qquad 0 < p < +\infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in [-1,1]} |f(t)|.$$

Of course, when $0 < p < 1$, $\|\cdot\|_p$ is not a norm; nevertheless, we keep this notation for convenience. We will also use the notations $\|\cdot\|_{d\alpha,p}$ and $\|\cdot\|_{w,p}$, defined by

$$\|f\|_{d\alpha,p} = \left(\int_{-1}^{1} |f|^p\,d\alpha\right)^{1/p} \qquad \text{or} \qquad \|f\|_{w,p} = \left(\int_{-1}^{1} |f|^p\,w\,dt\right)^{1/p}, \tag{2.5}$$

even for $0 < p < 1$.

Let w be a nonnegative function. We will call w a generalized Jacobi weight function ($w \in GJ$), if it can be written as

$$w(x) = \prod_{i=0}^{r+1} |x - t_i|^{\Gamma_i} \tag{2.6}$$

for $x \in [-1, 1]$ and $w(x) = 0$ for $|x| > 1$. Note that w is not necessarily integrable. We shall call $d\alpha$ a generalized Jacobi distribution when $\alpha' = \psi w$, where $w \in GJ$ and w is integrable, ψ is a positive continuous function in $[-1, 1]$ and the modulus of continuity ω of ψ satisfies

$$\int_0^1 \frac{\omega(t)}{t}\,dt < +\infty.$$

Sometimes we shall write $\Gamma_i(w)$ or $\Gamma_i(d\alpha)$ in place of Γ_i to indicate that they are parameters of w or $d\alpha$ respectively. Orthogonal polynomials corresponding to generalized Jacobi distributions are called generalized Jacobi polynomials. When $\Gamma_i = 0$, $1 \leqslant i \leqslant r$, and $\psi = 1$, "generalized Jacobi" reduces to "Jacobi."

Throughout this paper, we will use letters $c, c_1, c_2, \ldots$, etc. to denote constants depending only on weight functions and other fixed parameters involved, but their values may be different at different occurrences, even within the same formula. The notation $A \sim B$ means $|A^{-1}B| \leqslant c$ and $|AB^{-1}| \leqslant c$.

In the following, we list those properties of the generalized Jacobi polynomials that will be used in this paper. For the proof of these properties and the extensive study of the generalized Jacobi polynomials, see [2, 14]. For $w \in GJ$ in the form of (2.6) we define

$$w_n(x) = \left(\sqrt{1-x} + \frac{1}{n}\right)^{2\Gamma_0} \prod_{i=1}^{r} \left(|x - t_i| + \frac{1}{n}\right)^{\Gamma_i} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2\Gamma_{r+1}}. \quad (2.7)$$

For $d\alpha$ being a generalized Jacobi distribution, we also denote the corresponding one for $\alpha' = \psi w$ as $\alpha'_n = \psi w_n$.

LEMMA 2.1. *Let $d\alpha$ be a generalized Jacobi distribution. Then for every positive integer n*

$$|p_n(d\alpha, x)| \leqslant c\alpha'_n(x)^{-1/2} \left(\sqrt{1-x^2} + \frac{1}{n}\right)^{-1/2} \quad (2.8)$$

uniformly for $-1 \leqslant x \leqslant 1$ [2, Theorem 1.1, p. 226], in particular

$$|p_n(d\alpha, x)| \leqslant c[1 + (\alpha'(x)\sqrt{1-x^2})^{-1/2}] \quad (2.9)$$

uniformly for $-1 \leqslant x \leqslant 1$, and [14, Theorem 6.3.28, p. 120, and 9.22, p. 166]

$$\lambda_n(d\alpha, x) \sim \frac{1}{n} \alpha'_n(x) \left(\sqrt{1-x} + \frac{1}{n}\right)\left(\sqrt{1+x} + \frac{1}{n}\right) \quad (2.10)$$

uniformly for $-1 \leqslant x \leqslant 1$, in particular

$$\lambda_{kn}(d\alpha) \sim \frac{1}{n} (1 - x_{kn})^{\Gamma_0(d\alpha) + 1/2} \prod_{i=1}^{r} \left(|t_i - x_{kn}| + \frac{1}{n}\right)^{\Gamma_i(d\alpha)} \times (1 + x_{kn})^{\Gamma_{r+1}(d\alpha) + 1/2} \quad (2.11)$$

uniformly for $1 \leqslant k \leqslant n$, where $x_{kn} = x_{kn}(d\alpha)$, and [14, p. 170]

$$|p'_n(d\alpha, x_{kn})|^{-1} \sim \frac{1}{n} [\alpha'_n(x_{kn})]^{1/2} (1 - x_{kn}^2)^{3/4} \quad (2.12)$$

uniformly for $1 \leqslant k \leqslant n$.

Let $w \in GJ$, for a fixed $d > 0$, we define $\Delta_n(d)$ by

$$\Delta_n(d) = [-1 + dn^{-2}, 1 - dn^{-2}] \Big\backslash \bigcup_{i=1}^{r} [t_i - dn^{-1}, t_i + dn^{-1}].$$

We shall use χ_E to denote the characteristic function of a set E.

LEMMA 2.2 [14, Theorem 6.3.28, p. 120]. *Let $w \in GJ$ be integrable. Then for each $0 < p < +\infty$ there exists $d = d(p) > 0$ such that for every $R \in \Pi_n$,*

$$\|R\|_{w,p} \leqslant c\, \|R\chi_{\Delta_n(d)}\|_{w,p}.$$

3. GENERAL INEQUALITIES

In this section, we shall prove the inequalities that will be applied in the proof of the mean convergence of the generalized Jacobi series. These inequalities are of the general nature, and are related to the Hardy inequality and the Hilbert transform theorem. For general weight functions, they have been studied by several authors, see, for example, [13, and Refs. therein]. However, with the general weight functions, the conditions under which these inequalities hold are usually too general to check, thus inappropriate for our purpose. We shall prove these inequalities here for our special weight functions under simple conditions.

Throughout this paper, the singular integrals are to be taken in the principal value sense. For $p > 1$, we always use the notation $q = p/(p-1)$.

First we state three basic lemmas taken from [12].

LEMMA 3.1 [12, Lemma 3, p. 438]. *Let $1 < p < +\infty$, $r \geqslant R$, $R < -1$, $s \leqslant S$, and $s < -1$. Then*

$$\int_0^{+\infty} x^r(1+x)^{s-r} \left| \int_0^x g(y)\, dy \right|^p dx \leqslant c \int_0^{+\infty} x^p x^R (1+x)^{S-R}\, |g(x)|^p\, dx.$$

LEMMA 3.2 [12, Lemma 4, p. 438]. *Let $1 < p < +\infty$, $r \geqslant R$, $r > -1$, $s \leqslant S$, and $S > -1$. Then*

$$\int_0^{+\infty} x^r(1+x)^{s-r} \left| \int_x^{+\infty} g(y)\, dy \right|^p dx \leqslant c \int_0^{+\infty} x^p x^R (1+x)^{S-R}\, |g(x)|^p\, dx.$$

LEMMA 3.3 [12, Lemma 8, p. 440]. *Let $1 < p < +\infty$, $r > -1/p$, $s < 1 - 1/p$, $R < 1 - 1/p$, $S > -1/p$, $r \geqslant R$, and $s \leqslant S$. Then*

$$\int_0^{+\infty} \left| \int_0^{+\infty} \frac{g(y)}{x-y}\, x^r(1+x)^{s-r}\, dy \right|^p dx \leqslant c \int_0^{+\infty} |g(y)\, y^R(1+y)^{S-R}|^p\, dy.$$

LEMMA 3.4. *Let $1 < p < +\infty$, $rp > -1$, $sq < 1$, and $r \geqslant s$. Then*

$$\int_0^A \left| \int_0^B \frac{g(y)}{x+y}\, dy \right|^p x^{rp}\, dx \leqslant c \int_0^B |g(x)|^p x^{sp}\, dx,$$

where $c = c_1 A^{1+rp} B^{-sp-1}(1 + A^p/B^p)$.

Proof. Changing variables $x = Au/(1+u)$ and $y = Bv/(1+v)$ leads to

$$\begin{aligned}
&\int_0^A \left| \int_0^B \frac{g(y)}{x+y}\, dy \right|^p x^{rp}\, dx \\
&\quad = A^{1+rp} B^p \int_0^{+\infty} \left| \int_0^{+\infty} \frac{g(y)}{Au(1+v) + Bv(1+u)} \frac{1+u}{1+v}\, dv \right|^p \\
&\qquad \times \left(\frac{u}{1+u} \right)^{rp} \frac{du}{(1+u)^2} \\
&\quad \leqslant A^{1+rp-p} B^p \int_0^{+\infty} \left[\int_0^u \frac{|g(y)|}{u(1+v)} \frac{1+u}{1+v}\, dv \right]^p \left(\frac{u}{1+u} \right)^{rp} \frac{du}{(1+u)^2} \\
&\qquad + A^{1+rp} \int_0^{+\infty} \left[\int_u^{+\infty} \frac{|g(y)|}{v(1+u)} \frac{1+u}{1+v}\, dv \right]^p \left(\frac{u}{1+u} \right)^{rp} \frac{du}{(1+u)^2}.
\end{aligned}$$

For the last two terms, we apply Lemma 3.1 to the first term with $rp - p$, -2, $sp - p$, and $p - 2$ in place of r, s, R, and S, and apply Lemma 3.2 to the second term with rp, -2, sp, and $p - 2$ in place of r, s, R, and S. The conditions of these two lemmas are satisfied under the conditions of the present lemma. Thus we have

$$\begin{aligned}
&\int_0^A \left| \int_0^B \frac{g(y)}{x+y}\, dy \right|^p x^{rp}\, dx \\
&\quad \leqslant c[A^{1+rp-p} B^p + A^{rp+1}] \int_0^{+\infty} |g(y)|^p v^{sp}(1+v)^{-2-sp}\, dv \\
&\quad = c[A^{1+rp} B^{-1-sp}(1 + A^p/B^p)] \int_0^B |g(y)|^p y^{sp}\, dy. \quad \blacksquare
\end{aligned}$$

LEMMA 3.5. *Let $1 < p < +\infty$. Let U and V be generalized Jacobi weight functions. Then*

$$\int_{-1}^1 \left| \int_{-1}^1 \frac{g(y)}{x-y}\, dy \right|^p U^p(x)\, dx \leqslant c \int_{-1}^1 |g(x)|^p V^p(x)\, dx, \tag{3.1}$$

if

$$U^p \in L^1, \qquad V^{-q} \in L^1, \qquad \textit{and} \qquad U(x) \leqslant cV(x), \tag{3.2}$$

or, equivalently, if

$$\Gamma_i(U) > -1/p,\ \Gamma_i(V) < 1/q,\ \textit{and}\ \Gamma_i(V) \leqslant \Gamma_i(U)\ (0 \leqslant i \leqslant r+1). \tag{3.3}$$

Proof. To simplify the notation, rewrite $\Gamma_i(U) = \Gamma_i$, $\Gamma_i(V) = \gamma_i$, $0 \leqslant i \leqslant r+1$. Since $U(x) \sim |x-t_i|^{\Gamma_i}\,|x-t_{i+1}|^{\Gamma_{i+1}}$ for $t_i \leqslant x \leqslant t_{i+1}$, we have

$$\begin{aligned}
&\int_{-1}^{1} \left| \int_{-1}^{1} \frac{g(y)}{x-y}\,dy \right|^p U^p(x)\,dx \\
&\quad = \sum_{i=0}^{r} \int_{t_i}^{t_{i+1}} \left| \int_{-1}^{1} \frac{g(y)}{x-y}\,dy \right|^p U^p(x)\,dx \\
&\quad \leqslant c \sum_{i=0}^{r} \int_{t_i}^{t_{i+1}} \left| \int_{-1}^{1} \frac{g(y)}{x-y}\,dy \right|^p |x-t_i|^{p\Gamma_i}\,|x-t_{i+1}|^{p\Gamma_{i+1}}\,dx. \qquad (3.4)
\end{aligned}$$

For each i, we then break the inner integral of the last expression into three integrals over $(-1, t_i)$, (t_i, t_{i+1}), and $(t_{i+1}, 1)$, respectively, we shall estimate the corresponding terms separately. First, by changing variables

$$x - t_i = \frac{t_{i+1}-t_i}{1+X}, \qquad y - t_i = \frac{t_{i+1}-t_i}{1+Y},$$

we have

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \left| \int_{t_i}^{t_{i+1}} \frac{g(y)}{x-y}\,dy \right|^p |x-t_i|^{p\Gamma_i}\,|x-t_{i+1}|^{p\Gamma_{i+1}}\,dx \\
&\quad = \left(\frac{t_{i+1}-t_i}{2}\right)^{p\Gamma_i + p\Gamma_{i+1}+1} \int_0^{+\infty} \left| \int_0^{+\infty} \frac{g(y)}{X-Y}\,\frac{X+1}{Y+1}\,dY \right|^p \\
&\qquad \times \left(\frac{1}{X+1}\right)^{p\Gamma_i} \left(\frac{X}{X+1}\right)^{p\Gamma_{i+1}} \frac{dX}{(X+1)^2},
\end{aligned}$$

which, by Lemma 3.3 with $r = \Gamma_{i+1}$, $s = 1 - \Gamma_i - 2/p$, $R = \gamma_{i+1}$, and $S = 1 - \gamma_i - 2/p$, is bounded by

$$\begin{aligned}
&c \int_0^{+\infty} |g(y)|^p\,(1+Y)^{p\gamma_i - p\gamma_{i+1} - 2}\,Y^{p\gamma_{i+1}}\,dY \\
&\quad = c_1 \int_{t_i}^{t_{i+1}} |g(y)|^p\,|y-t_i|^{p\gamma_i}\,|y-t_{i+1}|^{p\gamma_{i+1}}\,dy
\end{aligned}$$

under the conditions (3.2). Thus we have obtained

$$\int_{t_i}^{t_{i+1}} \left| \int_{t_i}^{t_{i+1}} \frac{g(y)}{x-y}\, dy \right|^p U^p(x)\, dx \leqslant c \int_{t_i}^{t_{i+1}} |g(y)|^p V^p(y)\, dy. \tag{3.5}$$

Second, let $\bar{t} = (t_i + t_{i+1})/2$, we have

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} \left| \int_{-1}^{t_i} \frac{g(y)}{x-y}\, dy \right|^p |x-t_i|^{p\Gamma_i} |x-t_{i+1}|^{p\Gamma_{i+1}}\, dx \\
&\quad \leqslant c \int_{t_i}^{t_{i+1}} \left[\int_{-1}^{t_i} |g(y)|\, dy \right]^p |x-t_{i+1}|^{p\Gamma_{i+1}}\, dx \\
&\quad\quad + c \int_{t_i}^{t_i} \left[\int_{-1}^{t_{i-1}} |g(y)|\, dy \right]^p |x-t_i|^{p\Gamma_i}\, dx \\
&\quad\quad + c \int_{t_i}^{t_i} \left| \int_{t_{i-1}}^{t_i} \frac{g(y)}{x-y}\, dy \right|^p |x-t_i|^{p\Gamma_i}\, dx.
\end{aligned}$$

By Hölder inequality, the first term of the above is bounded by

$$\begin{aligned}
&c \int_{-1}^{t_i} |g(y)|^p V^p(y)\, dy \left[\int_{-1}^{t_i} [V(y)]^{-q}\, dy \right]^{p/q} \int_{t_i}^{t_{i+1}} |x-t_{i+1}|^{p\Gamma_{i+1}}\, dx \\
&\quad \leqslant c_1 \int_{-1}^{t_i} |g(y)|^p V^p(y)\, dy
\end{aligned}$$

under the conditions (3.2). The second term can be estimated similarly. Upon changing variables $x - t_i = u$, $y - t_i = -v$, the third term becomes

$$c_1 \int_0^{(t_{i+1}-t_i)/2} \left| \int_0^{(t_i - t_{i-1})/2} \frac{g(y)}{u+v}\, dv \right|^p u^{p\Gamma_i}\, du$$

which, by Lemma 3.4, is bounded by

$$c \int_0^{(t_i - t_{i-1})/2} |g(y)|^p v^{p\gamma_i}\, dv = c \int_{t_{i-1}}^{t_i} |g(y)|^p |y-t_i|^{p\gamma_i}\, dy$$

under the conditions (3.2). Thus we have proved

$$\int_{t_i}^{t_{i+1}} \left| \int_{-1}^{t_i} \frac{g(y)}{x-y}\, dy \right|^p U^p(x)\, dx \leqslant c \int_{-1}^{t_i} |g(y)|^p V^p(y)\, dy.$$

Similarly, we can prove that under the conditions (3.2)

$$\int_{t_i}^{t_{i+1}} \left| \int_{t_{i+1}}^{1} \frac{g(y)}{x-y}\, dy \right|^p U^p(x)\, dx \leqslant c \int_{t_{i+1}}^{1} |g(y)|^p V^p(y)\, dy.$$

The desired inequality (3.1) follows from the above two inequalities, (3.4) and (3.5). ▮

LEMMA 3.6. *Let* $1 < p < +\infty$, *and* $s < 1/q$. *Then for* $T > 1$

$$\int_0^T \left| \int_0^1 \frac{g(y)}{x+y}\, dy \right|^p (1+x)^{rp}\, dx \leqslant c\theta_r(T) \int_0^1 |g(y)|^p (1-y)^{sp}\, dy, \tag{3.6}$$

where

$$\theta_r(T) = \begin{cases} T^{rp-p+1}, & \text{if } rp - p > -1, \\ \log T, & \text{if } rp - p = -1, \\ 1, & \text{if } rp - p < -1. \end{cases} \tag{3.7}$$

Proof. We write the left hand side of (3.6) as two terms by breaking the outer integral into two integrals over $[0, 1]$ and $[1, T]$, respectively. The first term is bounded by

$$\int_0^1 \left| \int_0^1 \frac{g(y)}{x+y}\, dy \right|^p dx \leqslant c \int_0^1 |g(y)|^p (1-y)^{sp}\, dy$$

under the condition $s < 1/q$, which can be proved by first changing the variables $X + 1 = 1/(1-x)$, $Y + 1 = 1/(1-y)$, and then applying Lemmas 3.1 and 3.2 in exactly the same way as in the proof of Lemma 3.4. The second term is bounded by

$$c \left| \int_0^1 |g(y)|\, dy \right|^p \int_1^T (1+x)^{rp-p}\, dx,$$

which, by the Hölder inequality, is bounded by

$$c_1 \int_0^1 |g(y)|^p (1-y)^{sp}\, dy \left(\int_0^1 (1-y)^{-sq}\, dy \right)^{p/q} \int_0^T (1+x)^{rp-p}\, dx$$

$$\leqslant c\theta_r(T) \int_0^1 |g(y)|^p (1-y)^{sp}\, dy,$$

since $s < 1/q$. The proof is completed. ▮

4. THE GENERALIZED JACOBI SERIES

Let $d\alpha$ be a generalized Jacobi distribution. Let $S_n(d\alpha, f)$ be the partial sum of the generalized Jacobi series, i.e.,

$$S_n(d\alpha, f, x) = \sum_{k=0}^{n-1} c_k(f)\, p_k(d\alpha, x), \tag{4.1}$$

where

$$c_k(f)=\int_{-1}^{1} f(x)\, p_k(d\alpha, x)\, d\alpha. \tag{4.2}$$

Our main results on the mean convergence of the generalized Jacobi series are the following.

THEOREM 4.1. *Let $d\alpha$ be a generalized Jacobi distribution, and let u and w be generalized Jacobi weight functions. Let $1<p<+\infty$. Then*

$$\|S_n(d\alpha, f)\, w\|_{d\alpha, p} \leqslant c\, \|fu\|_{d\alpha, p} \tag{4.3}$$

for every f such that $\|fu\|_{d\alpha, p}<+\infty$ if and only if

$$\begin{aligned} w^p\alpha' \in L^1, &\qquad u^{-q}\alpha' \in L^1, \\ w^p(\alpha' \sqrt{1-x^2})^{-p/2}\, \alpha' \in L^1, &\qquad u^{-q}(\alpha' \sqrt{1-x^2})^{-q/2}\, \alpha' \in L^1 \end{aligned} \tag{4.4}$$

and

$$w(x) \leqslant cu(x). \tag{4.5}$$

Remark 4.1. When $w=u$, this theorem is proved by Badkov [2]. Earlier, Muckenhoupt [11] proved this for the Jacobi series. For $w \neq u$ and the Jacobi series, see [28]. The conditions (4.4) are proved to be necessary for very general distributions in [10]. The proof of the necessity of (4.5) in the following is due to Paul Nevai.

Since $S_n(d\alpha, f)$ is a projection operator, from Theorem 4.1 and the Weierstrass theorem we obtain

COROLLARY 4.2. *Under the assumptions of Theorem* 1 *and conditions* (4.4) *and* (4.5),

$$\lim_{n\to\infty} \|(S_n(d\alpha, f)-f)\, w\|_{d\alpha, p}=0$$

for every f such that $\|fu\|_{d\alpha, p}<\infty$.

Proof of Theorem 4.1. It follows from (2.2), (4.1), and (4.2) that

$$S_n(d\alpha, f, x)=\int_{-1}^{1} K_n(d\alpha, x, y)\, f(y)\, d\alpha(y).$$

Let $q_n(x)$ denote the orthonormal polynomials associated with the distribution $(1-x^2)\,d\alpha(x)$, i.e., $q_n(x) = p_n((1-(\cdot)^2)\,d\alpha, x)$. Let

$$h_1(x, y) = p_n(d\alpha, x)\, p_n(d\alpha, y), \tag{4.6}$$

$$h_2(x, y) = \frac{(1-y^2)\, p_n(d\alpha, x)\, q_{n-1}(y)}{x-y}, \tag{4.7}$$

and

$$h_3(x, y) = h_2(y, x). \tag{4.8}$$

Then it follows from [19] that

$$K_n(d\alpha, x, y) = a_n h_1(x, y) + b_n h_2(x, y) + b_n h_3(x, y),$$

where the numbers a_n and b_n depend on $d\alpha$ and n. Since $\alpha' > 0$, a.e., it follows from [20] that $|a_n|$ and $|b_n|$ are bounded by a constant independent of n (cf. [19, pp. 358–430]). Therefore, to prove that (4.4) and (4.5) imply (4.3), it is sufficient to prove that they imply

$$\int_{-1}^{1} \left| \int_{-1}^{1} h_k(x, y)\, f(y)\, d\alpha(y) \right|^p w^p(x)\, d\alpha(x) \leqslant c\, \|fu\|_{d\alpha, p}^p \tag{4.9}$$

for $k = 1$, 2, and 3.

For $k = 1$, we apply the Hölder inequality and use (2.9) for the upper bound of $p_n(d\alpha)$. It follows easily that (4.9) is true under the condition (4.4). For $k = 2$, we first note that since $S_n(d\alpha, f)$ is a polynomial of degree $\leqslant n-1$, it is sufficient to prove that

$$\int_{-1}^{1} \left| \int_{-1}^{1} h_2(x, y)\, f(y)\, d\alpha(y) \right|^p w^p(x)\, \chi_{\Delta_n(d)}(x)\, d\alpha(x) \leqslant c\, \|fu\|_{d\alpha, p}^p \tag{4.10}$$

by Lemma 2.2.

From inequality (2.8) it follows that

$$|p_n(d\alpha, x)| \leqslant (\alpha'(x)\sqrt{1-x^2})^{-1/2}, \qquad x \in \Delta_n(d). \tag{4.11}$$

Let $\sigma = \{i : \Gamma_i(d\alpha) < 0,\ 1 \leqslant i \leqslant r\}$, and the set $\tau_n(d)$ be defined as

$$\tau_n(d) = [-1, 1] \Big\backslash \bigcup_{i \in \sigma} [t_i - dn^{-1}, t_i + dn^{-1}].$$

Since $\Gamma_i(d\alpha) > -1$, $0 \leqslant i \leqslant r+1$, it follows also from (2.8) that

$$|q_{n-1}(y)| \leqslant c(\alpha'(y)(1-y^2)^{3/2})^{-1/2}, \qquad y \in \tau_n(d).$$

Therefore, we have from (4.7) that

$$\int_{-1}^{1}\left|\int_{-1}^{1} h_2(x, y) f(y) \chi_{\tau_n(d)}(y)\, d\alpha(y)\right|^p w^p(x) \chi_{\Delta_n(d)}(x)\, d\alpha(x)$$
$$\leqslant c\int_{-1}^{1}\left|\int_{-1}^{1} \frac{f(y)\,\phi_n(y)(1-y^2)(\alpha'(y)(1-y^2)^{3/2})^{-1/2}}{x-y}\, d\alpha(y)\right|^p$$
$$\times (\alpha'(x)\sqrt{1-x^2})^{-p/2}\, w^p(x)\, d\alpha(x), \tag{4.12}$$

where $\phi_n(y)$ is bounded by a constant independent of n. We now apply Lemma 3.5 with

$$g = f\phi_n(1-y^2)^{1/4}(\alpha')^{1/2},$$
$$U = (\alpha' \sqrt{1-x^2})^{-1/2}\, w(\alpha')^{1/p},$$
$$V = (\alpha' \sqrt{1-x^2})^{1/2}\, u(\alpha')^{-1/q}(1-x^2)^{-1/2},$$

and conclude that (4.12) is bounded by $\|fu\|_{d\alpha,p}^p$ under the conditions (4.4) and (4.5). Next we shall show that

$$\int_{-1}^{1}\left|\int_{-1}^{1} h_2(x, y) f(y)(1-\chi_{\tau_n(d)}(y))\, d\alpha(y)\right|^p w^p(x) \chi_{\Delta_n(d)}(x)\, d\alpha(x)$$
$$\leqslant c\, \|fu\|_{d\alpha,p}^p. \tag{4.13}$$

By definitions of $\Delta_n(d)$ and $\tau_n(d)$, the left hand side of (4.13) is bounded by the sum of

$$I_{ij} = \int_{t_j+dn^{-1}}^{t_{j+1}-dn^{-1}}\left|\int_{t_i-dn^{-1}}^{t_i+dn^{-1}} h_2(x, y) f(y)\, d\alpha(y)\right|^p w^p(x)\, d\alpha(x)$$

for $0 \leqslant j \leqslant r$ and $i \in \sigma$, where for $j=0$ we replace t_j+dn^{-1} by $t_j+dn^{-2} = -1+dn^{-2}$, and for $j=r$ we replace $t_{j+1}-dn^{-1}$ by $t_{j+1}-dn^{-2} = 1-dn^{-2}$. Since there are at most $r(r+1)$ terms in this sum, to prove (4.13) it is sufficient to estimate each I_{ij} by $\|fu\|_{d\alpha,p}^p$.

From inequality (2.8), we have for $i \in \sigma$

$$|q_{n-1}(y)| \leqslant cn^{\Gamma_i(d\alpha)/2}, \qquad y \in [t_i - dn^{-1}, t_i + dn^{-1}].$$

Therefore by (4.7) and (4.11),

$$I_{ij} \leqslant cn^{p\Gamma_i(d\alpha)/2}\int_{t_j+dn^{-1}}^{t_{j+1}-dn^{-1}}\left|\int_{t_i-dn^{-1}}^{t_i+dn^{-1}} \frac{f(y)\,\phi_n(y)}{x-y}\, d\alpha(y)\right|^p$$
$$\times (\alpha'(x)\sqrt{1-x^2})^{-p/2}\, w^p(x)\, d\alpha(x),$$

where $\phi_n(y)$ is bounded by a constant independent of n. Thus, if $j \neq i$, $i-1$, then

$$I_{ij} \leqslant cn^{p\Gamma_i(d\alpha)/2} \int_{t_j+dn^{-1}}^{t_{j+1}-dn^{-1}} (\alpha'(x)\sqrt{1-x^2})^{-p/2}\, w^p(x)\, d\alpha(x)$$
$$\times \left[\int_{t_i-dn^{-1}}^{t_i+dn^{-1}} |f(y)|\, d\alpha(y)\right]^p$$
$$\leqslant cn^{p\Gamma_i(d\alpha)/2} \int_{t_i-dn^{-1}}^{t_i+dn^{-1}} |f|^p\, u^p\, d\alpha$$

following the Hölder inequality and conditions (4.4). Since $i \in \sigma$, $\Gamma_i(d\alpha) < 0$, this gives us the desired bound. For $j=i$, we first break the outer integral of I_{ii} as integrals over $[t_i+dn^{-1}, \bar{t}_i]$ and $[\bar{t}_i, t_{i+1}-dn^{-1}]$, respectively, where $\bar{t}_i=(t_i+t_{i+1})/2$. For the second term, the above method for $j \neq i$, $i-1$ can be applied to derive the desired bound. For the first term, we further break the inner integral to integrals over $[t_i-dn^{-1}, t_i]$ and $[t_i, t_i+dn^{-1}]$. The essential term is

$$n^{p\Gamma_i(d\alpha)/2} \int_{t_i+dn^{-1}}^{\bar{t}_i} \left| \int_{t_i}^{t_i+dn^{-1}} \frac{f(y)\,\phi_n(y)}{x-y}\, d\alpha(y)\right|^p$$
$$\times (x-t_i)^{-p\Gamma_i(d\alpha)/2+p\Gamma_i(w)+\Gamma_i(d\alpha)}\, dx,$$

which, by changing variables $x-t_i=dn^{-1}(1+X)$ and $y-t_i=dn^{-1}(1-Y)$, can be estimated by Lemma 3.6 with $r=-\Gamma_i(d\alpha)/2+\Gamma_i(w)+\Gamma_i(d\alpha)/p$, $s=\Gamma_i(u)-\Gamma_i(d\alpha)/q$ and $T=(\bar{t}_i-t_i)\,d^{-1}n-1$ as

$$\leqslant cn^{p\Gamma_i(d\alpha)/2-rp-1}\theta_r(T) \int_0^1 |f(y)\,\alpha'(y)|^p\,(1-Y)^{sp}\, dY$$
$$\leqslant cn^{p\Gamma_i(d\alpha)/2-rp+sp}\theta_r(n) \int_{t_i}^{t_i+dn^{-1}} |f(y)|^p\, u(y)^p\, d\alpha(y),$$

where $s<1/q$ is implied by $u^{-q}\alpha' \in L^1$ at (4.4). Note that (4.5) is equivalent to $\Gamma_i(w) \geqslant \Gamma_i(u)$, which implies that $\Gamma_i(d\alpha)/2-r+s \leqslant 0$. From this, $s<1/q$, and the definition of $\theta_r(n)$ at (3.7), it follows that

$$n^{p\Gamma_i(d\alpha)/2-rp+sp}\theta_r(n) \leqslant c.$$

Thus we have proved that I_{ii} is bounded by $\|fu\|^p_{d\alpha,p}$. Similarly one can estimate $I_{i,i-1}$. Therefore we have proved (4.13). The inequality (4.10), thus inequality (4.9) for $k=2$, now follows from (4.12) and (4.13).

For $k=3$, we use a dual argument and derive the desired bound from the case $k=2$. This argument does not depend on the fact that our weight functions are the generalized Jacobi ones. We refer to [28, p. 889] for the detail. Thus, the proof for the sufficient part is completed.

The necessity of (4.4) is proved in [10] for very general distributions. We now prove the necesity of the condition (4.5). Since $S_n f$ is a projector, it follows from (4.3) that $S_n f$ converges to f in $\|\cdot w\|_{d\alpha, p}$ norm. Therefore by Fatou's Lemma

$$\|fw\|_{d\alpha, p} \leqslant c\ \|fu\|_{d\alpha, p}$$

for every f such that $\|fu\|_{d\alpha, p} < \infty$. In particular, f can be taken as characteristic functions of intervals in $[-1, 1]$. Therefore, we obtain

$$(w\alpha')(x) \leqslant c(u\alpha')(x), \qquad \text{a.e.}$$

This leads to (4.5). ∎

Acknowledgments

The author thanks Paul Nevai for the proof of the necessity of (4.5), and the referees for helpful comments.

References

1. R. Askey, Mean convergence of orthogonal series and Lagrange interpolation, *Acta. Math. Acad. Sci. Hungar.* **23** (1972), 71–85.
2. V. Badkov, Convergence in the mean and almost everywhere of Fourier series in polynomials orthogonal on an interval, *Math. USSR-Sb.* **24** (1974), 223–256.
3. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, Berlin, 1987.
4. P. Erdős and E. Feldheim, Sur le mode de convergence pour l'interpolation de Lagrange, *C. R. Acad. Sci. Paris* **203** (1936), 913–915.
5. D. Leviatan, The behavior of the derivatives of the algebraic polynomials of best approximation, *J. Approx. Theory* **35** (1982), 169–176.
6. D. S. Lubinsky, A. Máté, and P. Nevai, Quadrature sums involving pth powers of polynomials, *SIAM J. Math. Anal.* **18** (1987), 531–544.
7. J. Marcinkiewicz, Sur l'interpolation (I), *Studia Math.* **6** (1936), 1–17.
8. J. Marcinkiewicz and A. Zygmund, Mean values of trigonometric polynomials, *Fund. Math.* **28** (1937), 131–166.
9. A. Máté and P. Nevai, Necessary conditions for mean convergence of Hermite–Fejér interpolation, *in* "P. Erdős 75th Birthday Celebration" (B. Bollobás, Ed.), pp. 461–475, Cambridge Univ. Press, Cambridge, 1990.
10. A. Máté, P. Nevai, and V. Totik, Necessary conditions for weighted mean convergence of Fourier series in orthogonal polynomials, *J. Approx. Theory* **46** (1986), 314–322.
11. B. Muckenhoupt, Mean convergence of Jacobi series, *Proc. Amer. Math. Soc.* **24** (1970), 306–310.
12. B. Muckenhoupt, Mean convergence of Hermite and Laguerre series, II, *Trans. Amer. Math. Soc.* **147** (1970), 433–460.

13. B. Muckenhoupt and R. Wheeden, Two weight function norm inequalities for the Hardy–Littlewood maximal function and the Hilbert transform, *Studia J. Math.* **55** (1976), 279–294.
14. P. Nevai, Orthogonal polynomials, *Mem. Amer. Math. Soc.* **18** (1979), 213.
15. P. Nevai, Mean convergence of Lagrange interpolation, III, *Trans. Amer. Math. Soc.* **282** (1984), 669–698.
16. P. Nevai, Bernstein's inequality in L^p for $0 < p < 1$, *J. Approx. Theory* **27** (1979), 239–243.
17. P. Nevai and P. Vértesi, Mean convergence of Hermite–Fejér interpolation, *J. Math. Anal. Appl.* **105** (1985), 26–58.
18. P. Nevai and Y. Xu, Mean convergence of Hermite interpolation, preprint.
19. H. Pollard, The mean convergence of orthogonal series, II, *Trans. Amer. Math. Soc.* **63** (1948), 355–367.
20. E. A. Rahmanov, On the asymptotics of the ratio of the orthogonal polynomials, II, *Math. USSR-Sb.* **46** (1983), 105–117.
21. J. Szabados and A. K. Varma, On higher order Hermite–Fejér interpolation in weighted L^p-metric, preprint.
22. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc., New York, 1967.
23. P. Vértesi, Hermite–Fejér interpolations of higher order, I, *Acta Math. Hungar.* **54** (1989), 135–152.
24. P. Vértesi, Hermite–Fejér interpolations of higher order, II, *Acta Math. Hungar.* **56** (1990), 369–380.
25. P. Vértesi and Y. Xu, Weighted L^p convergence of Hermite interpolation of higher order, *Acta Math. Hungar.* **59** (1992), 423–438.
26. P. Vértesi and Y. Xu, Order of mean convergence of Hermite–Fejér interpolation, *Studia Sci. Math. Hungar.* **24** (1989), 391–401.
27. P. Vértesi and Y. Xu, Truncated Hermite interpolation polynomials, *Studia Sci. Math. Hungar.*, to appear.
28. Y. Xu, On the Marcinkiewicz–Zygmund inequality, *in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 879–891, Academic Press, San Diego, 1991.
29. Y. Xu, Marcinkiewicz–Zygmund inequality with derivatives, *Approx. Theory Appl.* **8** (1991), 100–107.
30. A. Zygmund, "Trigonometric Series," Vol. 2, Cambridge Univ. Press, Cambridge, 1959.